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THE LERAY SPECTRAL SEQUENCE

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The purpose of this note is to construct a Leray-type spectral sequence for homotopy classes of maps of simplicial presheaves, both stably and unstably, for any morphism of Grothendieck sites. This spectral sequence specializes to the ordinary Leray spectral sequence in sheaf cohomology theory, but may also be used for generalized étale cohomology theories such as étale K-theory.

Introduction

Let $u:\mathscr{C}\to\mathscr{D}$ be a morphism of Grothendieck sites, with structure functor $u:\mathscr{D}\to\mathscr{C}$ and associated presheaf functors $u_*:\mathscr{C}^\wedge\to\mathscr{D}^\wedge$ and $u^s:\mathscr{D}^\wedge\to\mathscr{C}^\wedge$. Here, $\mathscr{C}^\wedge$ denotes the category of presheaves, or set-valued contravariant functors on $\mathscr{C}$, and the direct image functor u_* is defined by $u_*(F)=F\circ u$. By definition, u_* is required to be continuous in the sense that $u_*(F)$ is a sheaf on $\mathscr{D}$ if F is a sheaf on $\mathscr{C}$. The inverse image functor u^s is defined to be the left adjoint of u_*. Again, by definition, u^s is required to be left exact (see [1, p. 355]).

Let X be a pointed locally fibrant simplicial presheaf on the site $\mathscr{C}$. In the first section of this note, I shall construct a Leray spectral sequence, with

$$E_1^{s,t}=H^{2s-t}(\mathscr{D};R^su_*X)\quad \text{``}\Rightarrow\text{''}\quad [*_{\mathscr{C}},\Omega^{t-s}X]\quad \text{for } t\geq s.$$

The construction of the spectral sequence implies that $E_1^{s,t}=0$ for $t>2s$. $[*_{\mathscr{C}},\Omega^{t-s}X]_{\mathscr{C}}$ denotes morphisms from the terminal simplicial presheaf $*_{\mathscr{C}}$ to the iterated loop object $\Omega^{t-s}X$ in the homotopy category associated to the category of simplicial presheaves on $\mathscr{C}$; it can be regarded as a cohomology object $H^{t-s}(\mathscr{C};X)$ for $\mathscr{C}$ with coefficients in the simplicial presheaf X. R^su_*X denotes the sheaf on $\mathscr{D}$ which is associated to the presheaf defined by $d\mapsto[*_{u(d)},\Omega^sX|_{u(d)}]_{u(d)}$, where $\Omega^sX|_{u(d)}$ denotes the restriction of Ω^sX to the site $\mathscr{C}\downarrow u(d)$, $*_{u(d)}$ is the terminal simplicial presheaf on that site, and the indicated morphisms are in a homotopy

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category which is defined accordingly. The quotation marks above mean that the spectral sequence usually only converges if extra assumptions are placed on X (see [5, 10]). The spectral sequence is a Bousfield–Kan spectral sequence for a tower of fibrations [3] arising from the homotopy theory of simplicial presheaves. The reader who is unhappy with Bousfield–Kan indexing should observe that applying re-indexing trick of [10, 5.54] to the Leray spectral sequence yields a spectral sequence with

$$\mathscr{E}_2^{p,q} = H^p(\mathscr{D}, R^q u_* X) \quad \text{“}\Rightarrow\text{”} \quad [*_{\mathscr{C}}, \Omega^{p-q} X].$$

If X happens to be an Eilenberg–Mac Lane object of the form $K(A,n)$ for some sheaf of abelian groups A on $\mathscr{C}$ and $0 \le s \le n$, then $R^s u_* X$ is the sheaf associated to the presheaf $d \mapsto [*_{u(d)}, K(A, n-s)]_{u(d)} = H^{n-s}(\mathscr{C} \downarrow u(d); A|_{\mathscr{C} \downarrow u(d)})$. In other words, $R^s u_* K(A,n)$ is the higher direct image sheaf $R^{n-s} u_* A$. In this case, the spectral sequence constructed here is really a truncated version of the traditional Leray spectral sequence. The truncation works because of the fringing effect in the Bousfield–Kan spectral sequence. One gets the traditional Leray spectral sequence out of this construction by rolling the same tape for presheaves of spectra; this will be done in the second section of this paper.

I do not seriously address the question of finding applications of this spectral sequence in this paper. The construction was motivated in part by a question of Weibel's. He wanted a machine-theoretic method of showing that, if $\pi : Y \to X$ is a finite map of schemes, then the set of isomorphism classes of rank n vector bundles on X is isomorphic to the set of $\pi_* \mathrm{Gl}_n$-torsors for the Zariski topology on Y. From the point of view of the theory given here, the basic idea is to show that $R^0 \pi_* B\mathrm{Gl}_n$ is trivial; this can be achieved by elementary techniques [4, A.3]. It follows that, if $i : B\mathrm{Gl}_n \to GB\mathrm{Gl}_n$ is a weak equivalence on the Zariski site of Y from $B\mathrm{Gl}_n$ to a globally fibrant model $GB\mathrm{Gl}_n$, then the induced map $\pi_* i : \pi_* B\mathrm{Gl}_n \to \pi_* GB\mathrm{Gl}_n$ is a weak equivalence of simplicial presheaves on the Zariski site over X. Then one can show that the adjunction map induces an isomorphism $[*_X, \pi_* B\mathrm{Gl}_n] \cong [*_Y, B\mathrm{Gl}_n]$ of homotopy classes relative to the Zariski topologies on X and Y. To relate this to the question that was actually posed, one needs to know that $[*_X, B\mathrm{Gl}_n]$ may be identified with the set of Gl_n-torsors on X, and hence with the set of isomorphism classes of rank n vector bundles on X, but this is a consequence of (a) of [7, 1.4]; the same result implies that $[*_Y, \pi_* B\mathrm{Gl}_n]$ is isomorphic to the set of $\pi_* \mathrm{Gl}_n$-torsors on Y. The argument works in general: the obstruction to an isomorphism of the form $[*, BG] \cong [*, \pi_* BG]$ for a sheaf of groups G lies in the set $R^0 \pi_* BG$.

There should be applications of this spectral sequence to calculation of K-theories arising from various topologies. It is a consequence of the construction, for example, that the étale K-theory of a Noetherian scheme of finite Krull dimension may be computed using a strongly convergent spectral sequence involving Nisnevich cohomology groups (the convergence is a consequence of a result of Kato and Saito which gives a bounded Nisnevich cohomological dimension for such schemes – see [9]). The calculational advantage of doing so has yet to be determined, however.

1. The basic construction

With the notation above, suppose that F is a globally fibrant [5] simplicial presheaf on $\mathscr{C}$. Then I claim that there is an adjointness isomorphism of the form

$$[*_{\mathscr{D}}, u_*F]_{\mathscr{D}} \cong [*_{\mathscr{C}}, F]_{\mathscr{C}}.$$

In effect, u_*F is globally fibrant, and so there are isomorphisms

$$\begin{aligned}[*_{\mathscr{D}}, u_*F]_{\mathscr{D}} &\cong \pi_0 \,\mathbf{hom}(*_{\mathscr{D}}, u_*F) \cong \pi_0 \,\mathbf{hom}(u^s *_{\mathscr{D}}, F) \cong \pi_0 \,\mathbf{hom}(*_{\mathscr{C}}, F)\\ &\cong [*_{\mathscr{C}}, F]_{\mathscr{C}}.\end{aligned}$$

To see that u_*F is globally fibrant, it suffices to show that u^s preserves trivial cofibrations. But u^s preserves monomorphisms and trivial local fibrations, and commutes with Kan's Ex^∞ functor, all by exactness, and the claim is proved.

Suppose that X is a pointed simplicial presheaf on $\mathscr{C}$, and find a weak equivalence $\eta : X \to GX$ such that GX is globally fibrant. Then there are isomorphisms of the form

$$\begin{aligned}[*_{\mathscr{C}}, \Omega^n X]_{\mathscr{C}} &\cong [*_{\mathscr{C}}, \Omega^n GX]_{\mathscr{C}}\\ &\cong [*_{\mathscr{D}}, u_* \Omega^n GX]_{\mathscr{D}}\\ &\cong [*_{\mathscr{D}}, \Omega^n u_* GX]_{\mathscr{D}}.\end{aligned}$$

The first isomorphism is induced by η, the second comes from the claim just proved and the fact that globally fibrant objects are preserved by the loop functor, and the third reflects the observation that the loop functor commutes with the direct image functor.

The Leray spectral sequence for $[*_{\mathscr{C}}, \Omega^{t-s}X]_{\mathscr{C}}$ is the cohomological descent spectral sequence for $[*_{\mathscr{D}}, \Omega^{t-s}u_*GX]_{\mathscr{D}}$ which arises from the Postnikov tower construction for u_*GX (see [5, p. 78]). This spectral sequence has

$$E_1^{s,t} = H^{2s-t}(\mathscr{D}; \pi_s u_* GX^{\sim}) \quad \text{“}\Rightarrow\text{”} \quad [*_{\mathscr{D}}, \Omega^{t-s} u_* GX]_{\mathscr{D}} \cong [*_{\mathscr{C}}, \Omega^{t-s} X]_{\mathscr{C}},$$

where $\pi_s u_* GX^{\sim}$ is the sheaf associated to the presheaf defined by $d \mapsto \pi_s u_* GX(d)$ for objects d of $\mathscr{D}$. It remains to identify this presheaf. But there are isomorphisms

$$\begin{aligned}\pi_s u_* GX(d) &= \pi_s GX(u(d))\\ &\cong [*_{u(d)}, \Omega^s GX|_{u(d)}]_{u(d)}\\ &\cong [*_{u(d)}, \Omega^s X|_{u(d)}]_{u(d)},\end{aligned}$$

since restriction along the functor $\mathscr{C} \downarrow u(d) \to \mathscr{C}$ is an exact functor which preserves global fibrations and commutes with Ex^∞ (see [5, p. 80]). It follows that $\pi_s u_* GX^{\sim}$ is the sheaf $R^s u_* X$ as defined in the introduction. We have proved

Theorem 1.1. *Let $u : \mathscr{C} \to \mathscr{D}$ be a morphism of Grothendieck sites, with associated* *direct image functor u_*. Let X be a pointed locally fibrant simplicial presheaf on $\mathscr{C}$. Then there is a Leray-type spectral sequence, with*

$$E_1^{s,t} = H^{2s-t}(\mathscr{D}; R^s u_* X) \quad \text{“}\Rightarrow\text{”} \quad [*_{\mathscr{C}}, \Omega^{t-s} X]_{\mathscr{C}}.$$

$R^s u_* X$ is the sheaf associated to the presheaf defined by $d \mapsto [*_{u(d)}, \Omega^s X|_{u(d)}]_{u(d)}$. $E_1^{s,t} = 0$ if $t > 2s$. □

2. The stable case

Insofar as the results of the last section were dependent to a certain extent on the existence of Postnikov towers, ‘rolling the same tape’ requires us to produce a precise analogue of that construction for presheaves of spectra. In order to do this properly, we need to revive Kan's approach [8] to stable homotopy theory somewhat. I shall begin by giving a new description of his notion of suspension.

Consider the inclusion of Δ^n in Δ^{n+1} induced by the inclusion of ordinal numbers $d^{n+1} : \mathbf{n} \subset \mathbf{n+1}$, and let the number $n+1$ be a base point for Δ^{n+1}. Observe that any ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$ uniquely extends to an ordinal number map $\theta_* : \mathbf{n+1} \to \mathbf{m+1}$ such that $\theta_*(n+1) = m+1$. Furthermore, $\theta_* d^{n+1} = d^{m+1}\theta$. In other words, Δ^{n+1} is a perfectly good functorial (even geometrically obvious) candidate for the cone $C\Delta^n$ on Δ^n. The usual trick of realizing the construction works: the cone CY of a simplicial set Y may be defined to be the colimit

$$CY = \varinjlim_{\Delta^n \to Y} \Delta^{n+1}$$

indexed over the simplex category of Y. Then the pointed cone C_*X of a pointed simplicial set Y may be defined by the pushout diagram

$$\begin{array}{ccc} C\Delta^0 & \xrightarrow{C*} & CX \\ \downarrow & & \downarrow \\ * & \longrightarrow & C_*X \end{array}$$

One has, of course, collapsed the cone on the base point $*$ of X to a point.

The maps $d^{n+1} : \Delta^n \to \Delta^{n+1}$ induce a natural pointed inclusion $X \subset C_*X$. The pointed simplicial set C_*X/X may be identified with the *Kan suspension* ΣX of X [8] (he uses the notation SX, but this conflicts with the standard notation for the singular functor).

It is obvious that the realization $|C_*X|$ of the cone of a pointed simplicial set X is naturally homeomorphic to the topological cone $C_*|X|$ of the pointed space $|X|$, and that $|\Sigma X|$ is naturally homeomorphic to the topological suspension $\Sigma|X|$ of the space $|X|$.

Furthermore, a pointed map $\varphi : \Sigma X \to Y$ may be identified with a collection of pointed set maps (suspension homomorphisms) $\varphi_n : X_n \to Y_{n+1}$, $n \geq 0$ such that:

(i) $d_0 \ldots d_n(\varphi_n(x)) = *$, $d_{n+1}\varphi_n(x) = *$ for each $x \in X_n$, and

(ii) for each ordinal number map $\theta : \boldsymbol{m} \to \boldsymbol{n}$, the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\varphi_n} & Y_{n+1} \\ \big\downarrow{\scriptstyle \theta^*} & & \big\downarrow{\scriptstyle \theta_*^*} \\ X_m & \xrightarrow{\varphi_m} & Y_{m+1} \end{array} \tag{1}$$

commutes, where $\theta_* : \boldsymbol{m+1} \to \boldsymbol{n+1}$ is the unique ordinal number map which restricts to θ along the standard inclusion of $\boldsymbol{m}$ in $\boldsymbol{m+1}$ and sends $m+1$ to $n+1$.

I define a *Kan spectrum* to be a sequence of pointed simplicial sets X^n, $n \geq 0$, together with pointed simplicial set maps of the form $\Sigma X^n \xrightarrow{\sigma} X^{n+1}$, $n \geq 0$. Note that such an object is almost what Kan [8] would call a prespectrum (but see also [2]). he reserves the term spectrum for objects Y consisting of pointed sets $Y_{(n)}$, $n \in \mathbb{Z}$, together with (pointed) face maps $d_i : Y_{(n)} \to Y_{(n-1)}$, and degeneracies $s_j : Y_{(n)} \to Y_{(n+1)}$, $i, j \geq 0$, such that the usual simplicial identities hold and such that, for each $x \in Y_{(n)}$, $d_i(x) = *$ for i sufficiently large. In fact, Y is not so much a spectrum as a cell complex associated to a Kan spectrum; see the method of associating a spectrum to a prespectrum in [8].

A Kan spectrum X is said to be *strictly fibrant* if each X^n is a Kan complex, for $n \geq 0$. Let $Y^{[n]}$ denote the nth Postnikov section of a pointed Kan complex Y, and recall that $Y^{[n]}$ has the form $Y^{[n]} = Y/{\sim_\nu}$, where the simplices x, y satisfy $x \sim_n y$ if $\mathrm{sk}_n(x) = \mathrm{sk}_n(y)$. Here, $\mathrm{sk}_n(x)$ is the composition

$$\mathrm{sk}_n \Delta^r \subset \Delta^r \xrightarrow{x} Y,$$

where $x : \Delta^r \to Y$ is the map which classifies x. It is a trivial observation that the bonding map $\Sigma X^n \to X^{n+1}$ induces a map $\Sigma(X^n)^{[k]} \to (X^{n+1})^{[k+1]}$. One may therefore define, for each $n \in \mathbb{Z}$, the nth Postnikov section $X^{[n]}$ of the strictly fibrant Kan spectrum X to consist of the spaces $(X^r)^{[s]}$ where $s - r = n$, and with bonding maps $\Sigma(X^r)^{[s]} \to (X^{r+1})^{[s+1]}$ as defined above. There is a strict fibration $X \to X^{[n]}$ which induces isomorphisms in stable homotopy groups $\pi_s(X) \cong \pi_s(X^{[n]})$ for $s \leq n$; this fibration is induced by collapsing X levelwise with respect to the obvious equivalence relations. Note that $\pi_s(X^{[n]}) \cong 0$ for $s > n$. The map $X \to X^{[n]}$ factors through the map $X \to X^{[n+1]}$, and the resulting map $X^{[n+1]} \to X^{[n]}$ is a strict fibration whose fibre is an Eilenberg–Mac Lane spectrum of type $K(\pi_{n+1}(X), n+1)$. The 'tower'

$$\cdots \to X^{[n+1]} \to X^{[n]} \to X^{[n-1]} \to \cdots$$

is the (natural) Postnikov tower of the strictly fibrant Kan spectrum X.

Recall that, most commonly, a *spectrum* Y of pointed simplicial sets consists of pointed simplicial sets Y^n, $n \geq 0$, together with pointed maps of the form $S^1 \wedge Y^n \to Y^{n+1}$, again for $n \geq 0$. Here, S^1 is the pointed simplicial set $\Delta^1/\partial\Delta^1$. Let SSpectra denote the category of spectra in the pointed simplicial set category, let KanSpectra denote the category of Kan spectra, and let TopSpectra denote the category of spectra in the pointed topological space category. Then, following [2], [6] and [8], one finds that there are adjoint pairs of functors of the form

$$\text{SSpectra} \underset{S}{\overset{|\cdot|}{\rightleftarrows}} \text{TopSpectra} \underset{|\cdot|}{\overset{S}{\rightleftarrows}} \text{KanSpectra}$$

induced by the ordinary realization functor $|\cdot|$ and the singular functor S. The canonical maps associated to these adjunctions induce isomorphisms in stable homotopy groups in all cases. All functors in sight also take strict fibrations to strict fibrations, since the realization of a Kan fibration is a Serre fibration. An ordinary spectrum of pointed simplicial sets Y therefore has a natural Postnikov tower associated to it, namely the tower of strict fibrations consisting of the spectra $S|(S|Y|)^{[n]}|$ and the induced fibrations between them. To simplify the notation, write $Y^{[n]}$ for $S|(S|Y|)^{[n]}|$.

If Z is a presheaf of spectra on a Grothendieck site $\mathscr{D}$, then we can form its nth Postnikov section $Z^{[n]}$ and the tower of pointwise strict fibrations

$$\cdots \to Z^{[n+1]} \to Z^{[n]} \to Z^{[n-1]} \to \cdots, \tag{2}$$

just by using the naturality of the construction given above.

A pointwise strict fibration $p: X \to Y$ of presheaves of spectra is a map of presheaves of spectra such that each map $p: X(d) \to Y(d)$ on sections is a strict fibration of spectra of pointed simplicial sets. Suppose that F is the fibre of p, and let $\Gamma^*\Sigma^n S^0$ be the constant presheaf of spectra associated to the spectrum $\Sigma^n S^0$. Then there is a natural long exact sequence of the form

$$\cdots \to [\Gamma^*\Sigma^n S^0, F] \to [\Gamma^*\Sigma^n S^0, X] \to [\Gamma^*\Sigma^n S^0, Y] \to [\Gamma^*\Sigma^{n-1} S^0, F] \to \cdots, \tag{3}$$

where the square brackets denote morphisms in the stable homotopy category associated to the category of presheaves of spectra on $\mathscr{D}$ [6]. In effect, the fibre sequence $F \to X \to Y$ may be replaced up to natural weak equivalence by the fibre sequence $QF \to QX \to QY$ of presheaves of Ω-objects. Then, for example, there is an isomorphism

$$[\Gamma^*\Sigma^n S^0, F] \cong [*, \Omega^{n+k} QF^k],$$

where the thing on the right denotes morphisms in the homotopy category associated to simplicial presheaves on $\mathscr{D}$. The claim follows by pasting together long exact sequences of the form

$$\cdots \to [*, \Omega^{n+k} \Omega F^k] \to [*, \Omega^{n+k} QX^k] \to [*, \Omega^{n+k} QY^k] \to [*, \Omega^{n+k-1} QF^k] \to \cdots.$$

These are natural long exact sequences associated to local fibrations between locally fibrant simplicial presheaves [5].

The exact couple associated to the Postnikov tower (2) via long exact sequences of the form (3) gives one of the descent spectral sequences for $[\Gamma^*\Sigma^n S^0, Z]$. By definition, it has

$$\begin{aligned} E_1^{s,t} &= [\Gamma^*\Sigma^{t-s}S^0, K(\pi_s Z, s)] \\ &\cong H^{2s-t}(\mathcal{D}; \pi_s Z^{\sim}), \end{aligned} \tag{4}$$

and converges to $[\Gamma^*\Sigma^{t-s}S^0, Z]$ under good conditions. Here, t and s are both allowed to run through all of $\mathbb{Z}$. Note that $E_1^{s,t} = 0$ if $t > 2s$.

Now, in the notation of the introduction, let X be a presheaf of spectra on the Grothendieck site $\mathscr{C}$, and choose a strict weak equivalence $QX \to GX$, where GX is a fibrant presheaf of spectra on $\mathscr{C}$ [6, p. 747]. Then GX is an Ω-object such that each GX^n is a globally fibrant simplicial presheaf. It follows that each $u_* GX^n$ is globally fibrant. Furthermore, u^s preserves cofibrations, and so u_* preserves trivial global fibrations and hence weak equivalences between globally fibrant objects. Thus, $u_* GX$ is a fibrant presheaf of spectra on $\mathscr{D}$. Note also that there are isomorphisms

$$\begin{aligned} [\Gamma^*\Sigma^n S^0, u_* GX]_{\mathscr{D}} &\cong [u^s \Gamma^*\Sigma^n S^0, GX]_{\mathscr{C}} \\ &\cong [\Gamma^*\Sigma^n S^0, GX]_{\mathscr{C}} \\ &\cong [\Gamma^*\Sigma^n S^0, X]_{\mathscr{C}}, \end{aligned}$$

again since GX is fibrant, and since the constant presheaf functor commutes with base change. Finally, there are isomorphisms of the form

$$\begin{aligned} \pi_s u_* GX(d) &\cong \pi_s GX(u(d)) \\ &\cong [\Gamma^*\Sigma^s S^0, GX|_{u(d)}]_{u(d)} \\ &\cong [\Gamma^*\Sigma^s S^0, X|_{u(d)}]_{u(d)}. \end{aligned}$$

We have proved

Theorem 2.1. *Let $u : \mathscr{C} \to \mathscr{D}$ be a morphism of Grothendieck sites, with associated direct image functor u_*. Let X be a presheaf of spectra on $\mathscr{C}$. Then there is a Leray-type spectral sequence, with*

$$E_1^{s,t} = H^{2s-t}(\mathscr{D}; R^s u_* X) \quad \text{``}\Rightarrow\text{''} \quad [\Gamma^*\Sigma^{t-s}S^0, X]_{\mathscr{C}}, \quad s, t \in \mathbb{Z}.$$

$R^s u_* X$ is the sheaf associated to the presheaf defined by $d \mapsto [\Gamma^*\Sigma^s S^0, X|_{u(d)}]_{u(d)}$. $E_1^{s,t} = 0$ if $t > 2s$.

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